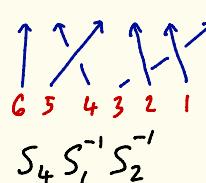


Work over  $\mathbb{k} = \mathbb{Q}(q)$  or maybe  $\mathbb{Q}(q, t)$ .  $z = q - q^{-1}$

$H_n$  = type A Hecke algebra

braid relation of course!



$\mathbb{k}$ -linear combinations of  $n$  strand braids plus quadratic relation:  $(S-q)(S+q^{-1})=0$

$$\begin{array}{ccc} & \Downarrow & \\ \text{skein relation: } & & S - S^{-1} = z \\ & \Updownarrow & \\ & \text{---} & z = \text{---} \end{array}$$

$AH_n$ : affine Hecke algebra — add invertible generators  $X_1, \dots, X_n$

$$\begin{array}{c} \bullet \\ \uparrow \end{array} \text{ plus relation } \begin{array}{c} \nearrow \\ \searrow \end{array} = \begin{array}{c} \searrow \\ \nearrow \end{array}$$

$$Z(AH_n) = \mathbb{k}[X_1^\pm, \dots, X_n^\pm]^{S_n} \hookrightarrow AH_n \quad \text{Bernstein center}$$

$$AH_n \rightarrow H_n, X_i \mapsto 1 \dots \begin{array}{c} \bullet \\ \uparrow \end{array} = \uparrow \text{ when on RH edge}$$

$$\text{Theorem (Dipper-James, Farnais-Graham)} \quad Z(AH_n) \rightarrow Z(H_n)$$

Basis for  $Z(H_n)$ :  $\{m_\lambda(x_1, \dots, x_n) \mid \lambda = \text{a partition of } n \text{ with first column removed}\}$

monomial symmetric polynomial  $x_1^{\lambda_1} \dots x_n^{\lambda_n} t(\text{sym})$

e.g.  $n=5$



$$x_1 + x_2 + x_3 + x_4 + x_5$$



Size = # partitions of  $n$  = #irreps =  $\dim Z(H_n)$

Corollary The maps  $\text{Sym} \xrightarrow{\Phi_n^+} Z(H_n)$ ,  $e_r \mapsto e_r(x_1^+, \dots, x_n^+)$   
 are asymptotically faithful

$\uparrow$   
Symmetric functions

$\uparrow$   
elementary

$$\bigcap_{n>0} \ker \Phi_n^+ = 0.$$

Conjecture The map  $\text{Sym} \otimes \text{Sym} \xrightarrow{\Phi_n^+ \otimes \Phi_n^-} Z(H_n)$  is asymptotically faithful too.

Difficult computationally to convert  $X_i^+$  into  $X_i^?$ 's ... Sage?

Rest of the talk: why??

Heisenberg category:  $\text{Heis}_k(z, t) \longleftrightarrow k \in \mathbb{Z}$  central charge

$k$ -linear strict monoidal category

Monoidal presentation Objects  $\wedge, \vee$   $\uparrow, \downarrow$  = identity endomorphisms  
 $\wedge \otimes \wedge \otimes \vee = \wedge \wedge \vee \dots$

Morphisms Braid relation, skein relation

$\uparrow$  Invertible, affine Hecke relation  $\cap, \cup$  Adjunction

Let  $\vdash \dashv$ .  $\cap = \uparrow = \dashv$

Case  $k = -2$

Declare  $\cup \cup$  is invertible

→ forces  $p_i q_i + 2 = q_i p_i$

One more relation:  $= \frac{t}{2} \wedge$  needs more definitions!

$$[\wedge \cup \dots]^{-1} = \begin{bmatrix} \wedge \\ \vdash \\ \vdash \\ \vdots \end{bmatrix}$$

AND SO ON!

Idea (Khovanov)

$\mathfrak{h}$  = Heisenberg Lie algebra

$p_n, q_m, c$

$[p_n, q_m] = \delta_{m,n} c$

$c$  central

Seek monoidal category

so  $K_0 \cong U(\mathfrak{h})$   
 $\langle c = k \rangle$

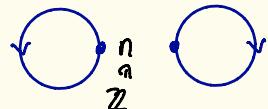
Grothendieck ring

$p_i = [\wedge], q_i = [\vee]$

$[p_i, q_j] = k$

Basic question now... bares for morphism spaces in  $\text{Heis}_k(z, t)$ ?

e.g. End(1)



Bubbles...

$S/\!/$

follows from earlier conjecture!

$\text{Sym} \otimes \text{Sym}$

The point:  $\text{Heis}_{-1}(z, t) \hookrightarrow \bigoplus_{n \geq 0} H_n\text{-mod}$

↑ acts by "induction"

↓ acts by "restriction"

$\text{Heis}_{-2}(z, t) \hookrightarrow$  replace  $H_n$  with level 2 cyclotomic Hecke algebras

⋮

To finish, I'll show you the case  $k=0$  — slightly easier!

In fact,  $\text{Heis}_0(z, t)$  is affinization of HOMFLY skein category (Turaev)

Let  $t = q^n$ .

$U_q(\mathfrak{gl}_n)$  Quantum group /  $\mathcal{Q}(q)$

$$E_i, F_i, D_i^{\pm} \quad [E_i, f_i] = \frac{D_i D_{i+1}^{-1} - D_{i+1}^{-1} D_i}{z}$$

$$\begin{matrix} i=1, \dots, n-1 & i=1, \dots, n \end{matrix}$$

↪   
  $V$  natural representation  
 of dimension  $n$

higher  
not  
elements

$$\left\{ \begin{array}{l} E_{ij} = E_{ir} E_{rj} - q^{-1} E_{rj} E_{ir} \quad i < r < j \\ F_{ij} = F_{rj} F_{ir} - q^{-1} F_{ir} F_{rj} \end{array} \right.$$

$H_{\text{crys}}(z, t)$

↑ acts as  $V \otimes -$   
(functor)

↓ acts as  $V^* \otimes -$

↪   
 ↗ acts as  $V \otimes V \otimes M \rightarrow V \otimes V \otimes M$ , R-matrix  $V \otimes V \circlearrowleft$ .  
(natural transformation)

$U_q(\mathfrak{gl}_n)\text{-mod}$  ↑ acts as  $V \otimes M \rightarrow M \otimes V \rightarrow V \otimes M$ , R-matrix squared.

○ =  $\frac{t-t^{-1}}{z} = \frac{q^n - q^{-n}}{q - q^{-1}} = [n]$  Quantum dimension of  $V$

$$\underset{\text{Heis}_q(\mathfrak{gl}_n)}{\text{End}}(\mathbb{H}) \rightarrow \text{End}(\text{Id}_{U_q(\mathfrak{gl}_n)\text{-mod}}) = \mathcal{Z}(U_q(\mathfrak{gl}_n)) \xrightarrow[\text{HC}]{} \mathbb{k}[x_1^{\pm}, \dots, x_n^{\pm}]^{S_n}$$

$$X_{ij} = z^{-\sum_{r=1}^{\min(i,j)} E_{r,i} D_r F_{rj} D_j} \quad (E_{i,i} = F_{i,i} = z^{-1})$$

$$\begin{aligned} \text{If } m > 0 \quad \textcircled{m} &\mapsto \sum_{i=1}^n q^{2i-n-1} \sum X_{i,i_1} X_{i,i_2} \dots X_{i_{m-1}, i_m} \\ &\mapsto \sum_{i_1 \leq \dots \leq i_m} \left(\frac{z}{q}\right)^{\#\{i_1, \dots, i_m\}-1} x_{i_1} \dots x_{i_m} \quad (\text{Modified complete symmetric function}) \end{aligned}$$

If  $m < 0$  Similar but  $\bar{x}^{-1}$  in place of  $x \dots$  Laurent symmetric functions.

The basis theorem follows ...

THE END